Algebra multilineare

V sp. vettoriale su $\mathbb{R}$ $\quad \dim V = n$

$\mathcal{B} = \{v_1, \dots, v_n\}$ base per V $\rightsquigarrow$ $\mathcal{B}^* = \{v^1, \dots, v^n\}$ BASE DUALE per V^*

$$v^i(v_j) = \delta_{ij}$$

$V \to V^{**}$ isomorfismo canonico $\quad v \longmapsto (v^* \longmapsto v^*(v))$

$V = V^{**}$

<u>Def</u>: $F: V_1 \times \dots \times V_k \to W$ è MULTILINEARE se è lineare in ogni componente

$\mathcal{B}_i = \{v_{i1}, \dots, v_{im_i}\}$ $\quad \mathcal{C} = \{w_1, \dots, w_h\}$ base di W

base di V_i

COORDINATE di F sono $\boxed{F^j_{j_1, \dots j_k}}$ univocamente determinati da

$$F(v_{1 j_1}, \dots, v_{k j_k}) = \sum_{i=1}^{h} \bigcirc \, w_i$$

Ex: F è determinata $\{F^{j}_{j_1, \dots, j_k}\}$

Cor: $\mathrm{Mult}(V_1, \dots, V_k; W) = \{F : V_1 \times \dots \times V_k \to W \text{ mult.}\}$

sp. vett. ha $\dim \mathrm{Mult}(V_1, \dots, V_k; W) = \dim V_1 \cdots \dim V_k \dim W$

Se $W = \mathbb{R}$ lo omettiamo:

$$\mathrm{Mult}(V_1, \dots, V_k) = \mathrm{Mult}(V_1, \dots, V_k; \mathbb{R})$$

Ex: $\mathrm{Mult}(V_1, \dots, V_k; W) \xrightarrow[\text{isom canonico}]{\sim} \mathrm{Mult}(V_1, \dots, V_k, W^*)$

$$F \longmapsto (v_1, \dots, v_k, w^*) \mapsto w^* F(v_1, \dots, v_k) \in \mathbb{R}$$

$$\mathrm{Hom}(V; W) \xrightarrow{\sim} \underbrace{\mathrm{Bil}(V, W^*)}_{= \mathrm{Mult}}$$

OPERAZIONI $\oplus$ e $\otimes$

$V_1, \dots, V_k$ sp. vett. $\leadsto$ $V_1 \oplus \dots \oplus V_k = \{(v_1, \dots, v_k) \mid v_i \in V_i\}$

$V_1 \otimes \cdots \otimes V_k := \underline{\mathrm{Mult}(V_1^*, \ldots, V_k^*)}$

$\dim(V_1 \oplus \cdots \oplus V_k) = \dim V_1 + \cdots + \dim V_k$

$\dim(V_1 \otimes \cdots \otimes V_k) = \dim V_1 \cdot \cdots \cdot \dim V_k$

$v_i \in V_i$

$\underline{v_1 \otimes \cdots \otimes v_k} \in V_1 \otimes \cdots \otimes V_k$ non sono tutti così!

$(v_1 \otimes \cdots \otimes v_k)(v^1, \ldots, v^k) = v^1(v_1) \cdot \cdots \cdot v^k(v_k) \in \mathbb{R}$

$\mathcal{B}_i = \{v_{i1}, \ldots, v_{im_i}\}$ base di V_i

Prop: $\{v_{1j_1} \otimes v_{2j_2} \otimes \cdots \otimes v_{kj_k}\}$ base di $\underline{V_1 \otimes \cdots \otimes V_k}$

dim: lin. indip.

Es: Base per $\mathbb{R}^2 \otimes \mathbb{R}^2$ $\quad \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix}$

Es 2.1.7: In $V \otimes W$ valgono:

$$(v+v') \otimes w = v \otimes w + v' \otimes w \qquad \forall v, v' \in V \quad \forall w \in W$$

$$\lambda (v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$$

$$v \otimes w = 0 \iff v = 0 \text{ oppure } w = 0$$

Es 2.1.8 $v, v' \in V$, $w, w' \in W$ non nulli

$$v, v' \text{ indip.} \Rightarrow v \otimes w \text{ e } v' \otimes w' \text{ indip.}$$

Es 2.1.9 v, v' indip. w, w' indip. $\Rightarrow$ $v \otimes w + v' \otimes w' \in V \otimes W$ non è $\bar{v} \otimes \bar{w}$ $\quad \bar{v} \in V$, $\bar{w} \in W$

PROPRIETÀ UNIVERSALE

$$\begin{tikzcd}
V_1 \times \dots \times V_k \arrow[r, "\pi"', "\text{multi}"] \arrow[dr, "F"', "\text{multi}"] & V_1 \otimes \dots \otimes V_k \arrow[d, dashed, "\exists!\, F' \text{ lineare}"] \\
 & \forall Z
\end{tikzcd}$$

multilineare (2.1.7)

$$\pi(v_1, \dots, v_k) = v_1 \otimes \dots \otimes v_k$$

dim: $\mathrm{Mult}(V_1, \dots, V_r; Z) \xleftarrow{\circ\pi} \mathrm{Hom}(V_1 \otimes \cdots \otimes V_r, Z)$

stessa dim. $+ (ex)$ è iniettiva $\square$

ISOMORFISMI CANONICI

Prop: $\exists$ no isom. canonici:

$V \oplus W \cong W \oplus V$ $\qquad (V \oplus W) \oplus Z \cong V \oplus W \oplus Z \cong V \oplus (W \oplus Z)$

$V \otimes W \cong W \otimes V$ $\qquad (V \otimes W) \otimes Z \cong V \otimes W \otimes Z \cong V \otimes (W \otimes Z)$

$V \otimes (W \oplus Z) \cong (V \otimes W) \oplus (V \otimes W)$

dim:

$\underset{\sim}{v} \otimes \underset{\sim}{w} \mapsto \underline{w} \otimes v \qquad v \in V \quad w \in W$

$$\begin{tikzcd}
V \otimes W \arrow[rr, dashed, "F'"] & & W \otimes V \\
& V \times W \arrow[ul, "\pi"] \arrow[ur, "F"'] &
\end{tikzcd}$$

$F(v, w) = w \otimes v$

Prop: $(V_1 \oplus \dots \oplus V_r)^* \cong V_1^* \oplus \dots \oplus V_r^*$

$(V_1 \otimes \dots \otimes V_r)^* \cong V_1^* \otimes \dots \otimes V_r^*$

Isom. canonico: $\mathrm{Hom}(V, W) = V^* \otimes W$

TENSORI

V sp. vettoriale su $\mathbb{R}$ dim $V = n$ $h, k \geq 0$ interi

(h, k)

$$\mathcal{T}_h^k(V) = \underbrace{V \otimes \cdots \otimes V}_{h} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k}$$

$T \in \mathcal{T}_h^k(V)$ TENSORE di tipo (h, k)

$$T : \underbrace{V^* \times \cdots \times V^*}_{h} \times \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R} \quad \text{mult.}$$

$(h,k) = (1,0)$ $\mathcal{T}_1^0(V) = V \ni T$ $T : V^* \to \mathbb{R}$ vettori $(1,0)$

$(0,1)$ $\mathcal{T}_0^1(V) = V^* \ni T$ $T : V \to \mathbb{R}$ covettori $(0,1)$

$(0,2)$ $\mathcal{T}_0^2(V) = V^* \otimes V^* \ni T$ $T : V \times V \to \mathbb{R}$ forme bilineari in V $(0,2)$

$(2,0)$ $\mathcal{T}_2^0(V) = V \otimes V$ $V^* \times V^* \to \mathbb{R}$ forme bilin in V^* $(2,0)$

$(1,1) \qquad \mathcal{T}_1^1(V) = V \otimes V^* \cong \mathrm{Hom}(V,V) \qquad T$ endomorfismi di V $(1,1)$

$$\mathcal{T}_1^1(V) = \mathrm{Hom}(V^*, V^*)$$

V^*

$(0,0) \qquad \mathcal{T}_0^0(V) = \mathbb{R}$ per convenzione scalari $(0,0)$

$$T \quad (h,k) \qquad T: \underbrace{V^* \times \dots \times V^*}_{h} \times \cancel{\underbrace{\overset{v_1 \, \dots \, v_k}{V \times \dots \times V}}_{k}} \longrightarrow \mathbb{R} \qquad \longleftarrow \text{mult}$$

può essere interpretata come

$$T: \underbrace{\overset{v_1 \, \dots \, v_k}{V \times \dots \times V}}_{k} \longrightarrow \underbrace{V \otimes \dots \otimes V}_{h} \qquad \longleftarrow \text{lin}$$

$T \ (1,k) \qquad T: V \times \dots \times V \longrightarrow V$ multilineare $\qquad V \times \dots \times V \to V \qquad (1,k)$

<u>Esempio</u>: Il prodotto vettoriale $\times$ in $\mathbb{R}^3$ $\qquad \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$
è tensore $(1,2)$

<u>Esempio</u>: $\det : \mathbb{R}^n \times \dots \times \mathbb{R}^n \to \mathbb{R} \qquad (0,n)$

$$(v_1, \dots, v_n) \longmapsto \det(v_1 \dots v_n)$$

COORDINATE

$$V \times V^* \to \mathbb{R}$$
$$(v, w) \mapsto w(v)$$

$\mathcal{T}_h^k(V)$ (h,k)

$\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ base di V $\rightsquigarrow$ $\mathcal{B}^* = \{\mathbf{v}^1, \dots, \mathbf{v}^n\}$ base di V^*

$$v^i(v_j) = \delta^i_j$$

Prop: Una base per $\mathcal{T}_h^k(V) = \underbrace{V \otimes \dots \otimes V}_{h} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{k}$

$$\underbrace{\{\mathbf{v}_{i_1} \otimes \dots \otimes \mathbf{v}_{i_h} \otimes \mathbf{v}^{j_1} \otimes \dots \otimes \mathbf{v}^{j_k}\}}$$

dim: Sono il numero giusto e sono indip.

$$\mathbf{T} \in \mathcal{T}_h^k(V) \quad \mathbf{T} = \boxed{T^{i_1 \dots i_h}_{j_1 \dots j_k}}\, \mathbf{v}_{i_1} \otimes \dots \otimes \mathbf{v}_{i_h} \otimes \mathbf{v}^{j_1} \otimes \dots \otimes \mathbf{v}^{j_k}$$

NOTAZIONE DI EINSTEIN

indici ripetuti si sommano da 1 a n.

Esempi: $\mathbf{v} \in V$ v^i le sue coordinate

$$\mathbf{v} = v^i \mathbf{v}_i$$

$T : V \to V$ endomorfismo $\qquad \underline{T(\mathbf{v}) = (\mathbf{w})} \qquad (1,1)$

$v^i \qquad v_j \qquad T^i_j \qquad T^i_j v^j = w^i$

g forma bilineare su $V \xrightarrow{(0,2)} g_{ij}$

$g(\mathbf{v}, \mathbf{w}) = \underline{v^i g_{ij} w^j} = g_{ij} v^i w^j$

CAMBIO BASE: $\mathcal{B} = \{v_1, \dots, v_n\} \leadsto \mathcal{B}^* = \{v^1, \dots, v^n\}$

$\mathcal{C} = \{w_1, \dots, w_n\} \leadsto \mathcal{C}^* = \{w^1, \dots, w^n\}$

Prop: $T \in \mathcal{T}^k_h(V)$. Le nuove coordinate $\hat{T}^{\dots}_{\dots}$ rispetto a $\mathcal{C}$ sono

$$\boxed{\hat{T}^{i_1 \dots i_h}_{j_1 \dots j_k} = B^{i_1}_{l_1} \cdots B^{i_h}_{l_h} A^{m_1}_{j_1} \cdots A^{m_k}_{j_k} T^{l_1 \dots l_h}_{m_1 \dots m_k}}$$

$$w_i = A_i^j v_j$$

A matrice di camb. base

$$v_1 = B_1^j w_j$$

$B = A^{-1}$ cioè

$AB = I = BA$

$$A_i^j B_j^k = \delta_i^k$$
$$A_i^j B_k^i = \delta_k^j$$

Prop: $w^i = B_j^i v^j \qquad v^i = A_j^i w^j$

ALGEBRA TENSORIALE

$$\mathcal{T}_h^k(V) = \underbrace{V \otimes \dots \otimes V}_{h} \otimes \underbrace{V^* \otimes \dots \otimes V^*}_{k}$$

$$\mathcal{T}_h^k(V) \otimes \mathcal{T}_{h'}^{k'}(V) = \mathcal{T}_{h+h'}^{k+k'}(V)$$

$$V \otimes - \otimes V \otimes V^* - V^* \otimes V - V \otimes V^* \otimes - \otimes V^*$$

$$T \in \mathcal{T}_h^k(V) \quad U \in \mathcal{T}_{h'}^{k'}(V) \quad T \otimes U \in \mathcal{T}_{h+h'}^{k+k'}(V)$$

$\mathcal{T} = \bigoplus_{h,k \geq 0} \mathcal{T}_h^k(V)$ estendo $\otimes$ in modo distributivo

con $\otimes$ una ALGEBRA NON COMMUTATIVA

$v \otimes w \neq w \otimes v$

In coordinate il $\otimes$ è semplice:

$$(T \otimes U)^{i_1 \cdots i_{k+k'}}_{j_1 \cdots j_{h+h'}} = T^{i_1 \cdots i_k}_{j_1 \cdots j_h} \, U^{i_{k+1} \cdots i_{k+k'}}_{j_{h+1} \cdots j_{h+h'}}$$

h e k scambiati

CONTRAZIONE DI UN TENSORE

$$C : \mathcal{T}^k_h(V) \to \mathcal{T}^{k-1}_{h-1}(V) \qquad h, k \geq 1$$

$$\|$$

$$\underbrace{V \otimes \cdots \overset{V}{\bigcirc} \cdots \otimes V}_{h} \otimes \underbrace{V^* \otimes \overset{V^*}{\bigcirc} \otimes V^*}_{k}$$

$1 \leq i \leq h$, $1 \leq j \leq k$ } dipende da i, j

$$\|$$

$$\underbrace{\underbrace{V \otimes - \otimes V}_{h-1} \otimes \underbrace{V^* \otimes - \otimes V^*}_{k-1}}_{Z} \otimes V \otimes V^* \xrightarrow{C} \underbrace{V \otimes - \otimes V}_{h-1} \otimes \underbrace{V^* \otimes - \otimes V^*}_{k-1} = \mathcal{T}^{k-1}_{h-1}(V)$$

$$Z \otimes V \otimes V^* \xrightarrow{C} Z$$

$$\uparrow \pi \qquad \nearrow F$$

$$Z \times V \times V^*$$

$$F(z, v, v^*) = v^*(v) \cdot z$$

In coordinate: $T^{i_1 \dots i_h}_{j_1 \dots j_k} \rightsquigarrow C(T)^{i_1 \dots \ell \dots i_h}_{j_1 \dots \ell \dots j_k}$

h, k

$T^{i}_{jk} \rightsquigarrow T^{i}_{ik}$ $\quad T^{i}_{ji}$

$(1,2)$ $\quad (0,1)$ covettore

$T \in \mathcal{T}^1_1(V) = End(V)$ $\qquad$ In coord T^i_j

$C(T) \in \mathcal{T}^0_0(V) = \mathbb{R}$ $\qquad C(T) = T^i_i$ TRACCIA

$g \in \mathcal{T}^2_0(V)$ forma bil.

g_{ij} $\qquad$ La traccia di una forma bilineare non è ben definita!

$\sum_{i=1}^{n} g_{ii}$ non è utile

$\square$